BCS microscopic theory of superconductivity

Superconductivity occurs because phonons lead to attractive effective interactions between electrons. Because of this attraction, electrons form Cooper pairs, excitations consisting of two electrons and behaving like bosons.

Bosons lead to an effective attractive interaction

$$\hat{H}_{BCS} = \sum_{p} \xi_p \, \hat{a}_p^{+} \hat{a}_p - \frac{1}{2} \sum_{\substack{p+p' \\ =q+q'}} V_{pp'qq'} \, a_{p\alpha}^{+} a_{p'\beta}^{+} \, a_{q\beta} \, a_{q'\alpha}$$

Phonons result in attraction only sufficiently close to the Fermi surface, with ω_D being the characteristic scale

$$V_{pp'qq'} = \begin{cases} \lambda, & \xi_{max} < \omega_D \\ 0, & \xi_{max} > \omega_D \end{cases} \qquad \times \frac{1}{V} \text{ - volume}$$

$$\xi_{max} = max\left(|\xi_p|, |\xi_{p'}|, |\xi_q|, |\xi_{q'}| \right)$$

In a Cooper pair electrons have roughly opposite momenta. In other words, the momentum of a Cooper pair is significantly smaller than the momenta of constituent electrons.

For simplicity, consider pairing which couples electrons with opposite momenta.

$$\hat{H} = \sum_{p} \xi_p \, a_p^{+} a_p - \frac{\lambda}{2V} \sum_{p,q} a_{p\alpha}^{+} \, a_{-p\beta}^{+} \, a_{q\beta} \, a_{-q\alpha}$$

[illegible] $\hat{?}^{+} = \sum a^{+} a^{+}$ describes the

The operator $\hat{B}^+_{\alpha\beta} = \sum_p a^+_{p\alpha} a^+_{-p\alpha}$ describes the creation of a Cooper pair whose configuration of spins is $|\alpha\beta\rangle$. Then the interaction term has the form $-\frac{\lambda}{2V}\sum_{\alpha,\beta} \hat{B}^+_{\alpha\beta}\hat{B}_{\alpha\beta}$.

Find out what spin configurations are possible.

$$\psi^+_\alpha(r)\psi^+_\alpha(r) = 0 \rightarrow \int \psi^+_\alpha(r)\psi^+_\alpha(r)\,dr = 0$$

Use that $\psi^+_\alpha(r) = \sum_k e^{-ikr} a^+_k$

$$= \sum_{k'} e^{-ik'} a^+_{k'}$$

$$\int e^{-ir(k+k')}\,dr = V\delta_{k,-k'}$$

$$\rightarrow \sum_k a^+_{k\alpha} a^+_{-k\alpha} = 0 \leftrightarrow \hat{B}_{\uparrow\uparrow} = \hat{B}_{\downarrow\downarrow} = 0$$

Similarly, using that $\{\psi^+_\alpha(r), \psi^+_\alpha(\alpha)\} = 0$, we can show that $\hat{B}_{\uparrow\downarrow} = -\hat{B}_{\downarrow\uparrow}$

So, the interaction takes the form

$$-\frac{\lambda}{V}\sum_{p,q} a^+_{p\uparrow} a^+_{-p\downarrow} a_{q\downarrow} a_{-q\uparrow}$$

i.e. it couples opposite spins $\rightarrow$ The spin of a Cooper pair $= 0$

This is a consequence of choosing the interaction to be a spin-independent constant. In principle, when the interaction is spin-dependent, the spin of a Cooper pair can be 1 (= triplet superconductivity).

can be 1 (= triplet superconductivity).

Let's do mean field, to describe how a transition to a superconductive (= paired) state takes place.

Introduce the order parameter

$$\Delta = -\frac{\lambda}{V}\sum_p \langle a_{p\downarrow} a_{-p\uparrow}\rangle \quad , \quad \Delta^* = -\frac{\lambda}{V}\sum_p \langle a^+_{p\uparrow} a^+_{-p\downarrow}\rangle$$

$$\sum_{p,q} a^+_{p\uparrow} a^+_{-p\downarrow} a_{q\downarrow} a_{-q\uparrow} = \left[-\frac{V\Delta^*}{\lambda} + \sum_p a^+_{p\uparrow} a^+_{-p\downarrow} + \frac{V\Delta^*}{\lambda}\right)\left]\left[\frac{-\Delta V}{\lambda} + \sum_q a_{q\downarrow} a_{-q\uparrow} + \frac{\Delta V}{\lambda}\right)\right]$$

$$\rightarrow \; -\frac{V}{\lambda}\,\Delta\sum_p a^+_{p\uparrow} a^+_{-p\downarrow} + -\frac{V}{\lambda}\,\Delta^*\sum_q a_{q\downarrow} a_{-q\uparrow} - \frac{|\Delta|^2}{\lambda^2} V^2$$

$$\boxed{\hat{H}_{BCS} = \sum_p \Big[\xi_p (a^+_{p\uparrow} a_{p\uparrow} + a^+_{p\downarrow} a_{p\downarrow}) + \Delta\, a^+_{p\uparrow} a^+_{-p\downarrow} + \Delta^* a_{p\downarrow} a_{-p\uparrow}\Big] + \frac{|\Delta|^2}{\lambda} V}$$

$$\hat{H}_{BCS} = \sum_p \left[(a^+_{p\uparrow} \;\; a_{-p\downarrow}) \begin{pmatrix} \xi_p & \Delta \\ \Delta^* & -\xi_p \end{pmatrix} \begin{pmatrix} a_{p\uparrow} \\ a^+_{-p\downarrow} \end{pmatrix} + \xi_p \right] + \frac{|\Delta|^2}{\lambda} V$$

Does not matter (Δ-independ.)

The eigenvalues of the matrix are

$$\Lambda = \pm\left(\xi_p^2 + |\Delta|^2\right)^{\frac{1}{2}}$$

The diagonalised Ham-n has the form

$$\hat{H}_{BCS} = \text{const} + \sum_p (c^+_{p\uparrow} \;\; c_{-p\downarrow}) \begin{pmatrix} E_p & 0 \\ 0 & -E_p \end{pmatrix} \begin{pmatrix} c_{p\uparrow} \\ c^+_{-p\downarrow} \end{pmatrix} + \frac{|\Delta|^2}{\lambda} V$$

$$\hat{H}_{BCS} = const + \sum_p (c^+_{p\uparrow} \;\; c_{-p\downarrow}) \begin{pmatrix} & \\ 0 & -E_p \end{pmatrix} \begin{pmatrix} \\ c^+_{-p\downarrow} \end{pmatrix} + \frac{|\Delta|^2}{\lambda} V$$

$$= \sum_p \left[E_p \left(c^+_{p\uparrow} c_{p\uparrow} + c^+_{p\downarrow} c_{p\downarrow} \right) - E_p \right] + \frac{|\Delta|^2}{\lambda} V + const$$

$$\boxed{E_p = \sqrt{\xi_p^2 + |\Delta|^2}}$$ – Excitation dispersion

Note: all excitations have positive energies

It is possible to assume, without loss of generality, that $\Delta \in \mathbb{R}$ (the phases of states $p\uparrow$ and $-p\downarrow$ may be adjusted accordingly)

$$\begin{cases} c_{p\uparrow} = u_p a_{p\uparrow} + v_p a^+_{-p\downarrow} \\ c_{-p\downarrow} = v_p a^+_{p\uparrow} - u_p a_{-p\downarrow} \end{cases}$$ – Bogolyubov transformation

$$u_p = \left[\frac{1}{2} \left(1 + \frac{\xi_p}{\sqrt{\xi_p^2 + \Delta^2}} \right) \right]^{\frac{1}{2}}$$

$$v_p = \left[\frac{1}{2} \left(1 - \frac{\xi_p}{\sqrt{\xi_p^2 + \Delta^2}} \right) \right]^{\frac{1}{2}}$$

Consider very small Δ, smaller than the temperature T

$$E_p = \sqrt{\xi_p^2 + |\Delta|^2} \approx \xi_p + \frac{|\Delta|^2}{\xi_p}$$

The quadratic coefficient in the Ginzburg-Landau functional comes from

$$\frac{|\Delta|^2}{\lambda} V \text{ and } \sum_p \Big(\underset{\substack{\approx \\ -\xi_p - \frac{|\Delta|^2}{\xi_p}}}{-E_p} + \underbrace{E_p \left(c^+_{p\uparrow} c_{p\uparrow} + c^+_{p\downarrow} c_{p\downarrow} \right)}_{\text{Excitations which matter at energies}} \Big)$$

$$-\xi_p - \frac{|\Delta|^2}{2\xi_p}$$

Excitations which matter at energies $E_p \sim T$

$$F = F_0 + V\frac{|\Delta|^2}{\lambda} - |\Delta|^2 \sum_{\substack{p \\ |\xi_p| \gtrsim T \\ |\xi_p| < \omega_D}} \frac{1}{2\xi_p} + C_4 |\Delta|^4 + \dots$$

Computing the quadratic coefficient,

$$F = F_0 + V|\Delta|^2\left(\frac{1}{\lambda} - \underbrace{\nu_0}_{\text{DoS}} \ln\frac{\omega_D}{T}\right) + C_4|\Delta|^4 + \dots$$

$$T_c \sim \omega_D \, e^{-\frac{1}{\lambda\nu_0}}$$

$$\boxed{T_c \approx \frac{2\gamma}{\sqrt{\pi}}\, \omega_D \, e^{-\frac{1}{\lambda\nu_0}}}$$

γ - Euler constant

Superconductivity exists at $T=0$ for arbitrarily weak interactions (λ)

The quasiparticles with the dispersion $E_p = (\xi_p^2 + |\Delta|^2)^{\frac{1}{2}}$ exist on top of the ground state and have a gap of $|\Delta|$